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Antiunitary Symmetry Operators in Quantum Mechanics

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A criterion to decide that some symmetries of a quantum system must be realized as antiunitary operators is given. It is based on some mathematical theorems about the second cohomology group of the symmetry group when expressed in terms of those of a normal subgroup and the corresponding factor group. It is also shown that this criterion implies that the only possibility for the unitary subgroup in the Galilean case is that generated by the space reflection and the connected component containing the identity; otherwise only massless systems would arise.

1. INTRODUCTION

In the usual formulation of quantum mechanics in Hilbert spaces, every symmetry of a quantum system is realized as unitary or antiunitary operator, but the ray rather than vector character of the states shows that any two such operators differing through a numerical factor of modulus 1 must be considered as the same operator. In other words, symmetries of a quantum system must be realized as unitary or antiunitary projective transformations: consequently, a symmetry group G must be realized in the space of quantum states by means of a semiunitary projective representation (hereafter SUPR). The set of symmetries realized in a unitary way is a subgroup G_U of index 1 or 2 which is called the unitary subgroup. In

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particular, when G is a connected Lie group the continuity of the representation implies that $G_U = G$. But in the general case G_U is different from G and the computation of the SUPRs of the group G depends on the choice for the unitary group: more information about this question may be found in some of our previous papers (Cariñena and Santander 1975, 1979, 1980).

How is this mathematical problem related to physics? Sometimes, physical reasons may indicate which is the unitary subgroup in each case, but this ought to be properly done, with a minimum of assumptions, and we feel it worth spending a little time in the analysis of such a problem.

The connected component G_0 of a Lie group G is a normal subgroup; it will be contained in the unitary subgroup G_U for every possible choice for G_U. Each SUPR of G subduces a UPR of G_U and therefore one of G_0 (irreducibility may be lost in any of two steps). We will see that, depending on the choice for G_U, not every UPR of G_0 arises in such a process, and this fact may be enough to dismiss some of the possible choices for G_U.

In the following section a few mathematical theorems are given; they are straightforward generalizations of Mackey's well-known results (1958) about cohomology groups of semidirect product of groups when the action of G on $\mathbf{T}$ (the one-dimensional torus group) is nontrivial. Section 3 is devoted to show how these theorems, when properly used, lead in a natural way to the choice for G_U in the case of the complete Galilean group (otherwise the representations describing massive elementary particles would not arise). The case of kinematic groups is analyzed in Section 4.

2. THE MAIN THEOREMS

As we are going to deal with SUPRs of a group G with respect to a closed subgroup G_U of index 1 or 2, we must study their factor systems $\omega \in Z^2{}_*(G, \mathbf{T})$, where the action of G on $\mathbf{T}$ is given by $\lambda^g = \lambda$ if $g \in G_U$, and $\lambda^g = \lambda^*$ if $g \notin G_U$, the asterisk standing for complex conjugation.

There are some papers devoted to the study of how to determine a complete set of inequivalent factor systems of a group G through those of an invariant subgroup (van den Broek, 1976, and other references therein). In particular we will be interested in the case where the group G is a semidirect product group $G = H \odot V$. Then, every element of G can be expressed in only one way as a pair (a, α), with $a \in H$ and $\alpha \in V$. Elements of H will be denoted by $a, b, c \ldots$ and elements of V by $\alpha, \beta, \gamma, \ldots$. The composition law in G is $(a, \alpha)(b, \beta) = (ab^\alpha, \alpha\beta)$. The factor systems of G when the action of G on $\mathbf{T}$ is trivial have been studied by Mackey (1958) (see also Parthasarathy, 1969); a straightforward generalization of his results leads to the following theorem.

Theorem 1. Let ω' be a factor system of G with respect to the subgroup G_U and let H_U and V_U denote the intersections $H_U = H \cap G_U$, $V_U = V \cap G_U$, respectively: There exist an equivalent factor system $\omega \in Z^2{}_*(G, \mathbf{T})$ which decomposes as a product

$$\omega[(a,\alpha),(b,\beta)] = \xi(a, b^\alpha)[\eta(\alpha,\beta)]^{ab^\alpha}[\Lambda(b,\alpha)]^a \quad (2.1)$$

with $\xi \in Z^2{}_*(H, \mathbf{T})$, $\eta \in Z^2{}_*(V, \mathbf{T})$, while Λ: $H \times V \to \mathbf{T}$ is a Borel function such that

$$\xi(a^\alpha, b^\alpha) = [\xi(a,b)]^\alpha \cdot \frac{\Lambda(ab,\alpha)}{\Lambda(a,\alpha)[\Lambda(b,\alpha)]^{ab^\alpha}} \quad (2.2a)$$

$$\Lambda(a,\alpha\beta) = \Lambda(a^\beta,\alpha)[\Lambda(a,\beta)]^\alpha \frac{[\eta(\alpha,\beta)]^{a^{\alpha\beta}}}{\eta(\alpha,\beta)} \quad (2.2b)$$

In these formulas λ^α and λ^a stand for $\lambda^{(e,\alpha)}$ and $\lambda^{(a,\varepsilon)}$, respectively.

A detailed proof is given in Santander (1974). It follows the pattern of that of Parthasarathy (1969) and it will not be given here. It can also be done by a slight modification of the proof of Theorem 1 in the paper by van den Broek (1976). The function m in that paper is now $m(\alpha,\beta) = e$ because of the semidirect structure; attention must be payed to the particular choice for the section $r(\alpha)$.

Conversely, if actions of H and V on $\mathbf{T}$ are given, let G_U be the subgroup of index 1 or 2 generated by the kernels of ineffectiveness of each action:

Theorem 2. If $\xi \in Z^2{}_*(H, \mathbf{T})$, $\eta \in Z^2{}_*(V, \mathbf{T})$, and Λ is a Borel function satisfying the above relations (2.2), then ω defined by (2.1) is a cocycle $\omega \in Z^2{}_*(G, \mathbf{T})$.

The proof of this theorem is a cumbersome but straightforward calculation, so that it is omitted.

These two theorems are very useful in the case of a nonconnected Lie group which is a semidirect product group such that $G = G_0 \odot \pi_0(G)$, G_0 being the connected component of G. The normal subgroup G_0 is always contained in G_U and therefore in this case the decomposition (2.1) reduces to $[V = \pi_0(G)]$

$$\omega[(a,\alpha),(b,\beta)] = \xi(a, b^\alpha)\eta(\alpha,\beta)\Lambda(b,\alpha) \quad (2.3)$$

where Λ: $G\times V\to\mathbf{T}$ is a Borel function satisfying

$$\xi(a^{\alpha}, b^{\alpha}) = [\xi(a,b)]^{\alpha}\frac{\Lambda(ab,\alpha)}{\Lambda(a,\alpha)\Lambda(b,\alpha)} \tag{2.4a}$$

$$\Lambda(a,\alpha\beta) = \Lambda(a^{\beta},\alpha)[\Lambda(a,\beta)]^{\alpha} \tag{2.4b}$$

Once ξ and η have been chosen, how many solutions Λ do exist for these equations? It would be interesting to know a necessary condition for the existence of (at least) one such Λ. The first equation provides such a condition: every $\alpha\in V$ defines an application τ_{α}: $Z^2(G_0,\mathbf{T})\to Z^2(G_0,\mathbf{T})$, $(\tau_{\alpha}\xi)(a,b)=\xi(a^{\alpha^{-1}}, b^{\alpha^{-1}})$; this mapping is an endomorphism such that $\tau_{\alpha}[B^2(G_0,\mathbf{T})]\subset B^2(G_0,\mathbf{T})$, so that there is an induced homomorphism $\bar{\tau}_{\alpha}$: $H^2(G_0,\mathbf{T})\to H^2(G_0,\mathbf{T})$. With this notation the relation (2.4.a) may be rewritten as

$$\left[(\tau_{\alpha^{-1}}\xi)\cdot(\xi^{\alpha})^{-1}\right](a,b) = \frac{\Lambda(ab,\alpha)}{\Lambda(a,\alpha)\Lambda(b,\alpha)}$$

Therefore, in order that a solution may exist, the factor system $(\tau_{\alpha^{-1}}\xi)(\xi^{\alpha})^{-1}$ must be a trivial factor system. This fact is independent of the choice for η. The preceding results can be summarized in the following theorem.

Theorem 3. With the above conditions and notations, the existence of a class $\bar{\omega}$ such that its restriction to G_0 and $V=\pi_0(G)$ are $\bar{\xi}$ and $\bar{\eta}$, respectively, does not depend on $\bar{\eta}$. A necessary condition for the existence of such $\bar{\omega}$ is that

$$\bar{\tau}_{\alpha^{-1}}\bar{\xi} = \bar{\xi}^{\alpha}, \qquad \forall\alpha\in V$$

3. PARITY AND TIME REVERSAL IN "NONRELATIVISTIC" QUANTUM PHYSICS

Let us see how these mathematical theorems can be useful. So, we choose a highly well-known group, the complete Galilean group. When semiunitary projective representations of this group are considered, the unitary subgroup is chosen to be $\mathcal{G}_U=\mathcal{G}_0\cup I_S\mathcal{G}_0$, where I_S denotes space inversion. The support for this choice is the analogy to the relativistic case of the complete Poincaré group. We want to remark that there is an additional—and perhaps more compelling—reason: with a different choice for $\mathcal{G}_U$ the restriction of any irreducible SUPR of $(\mathcal{G},\mathcal{G}_U)$ to the connected component would correspond to a "massless representation" of $\mathcal{G}_0$. However these UPRs of $\mathcal{G}_0$ have been shown to be unphysical. Furthermore, what about massive systems?

In complete analogy to the structure of the complete Poincaré group, the complete Galilean group is a nonconnected Lie group with four connected components: That containing the identity is a (normal) subgroup $\mathcal{G}_0$ which is called the orthochronous proper Galilean group. We will use throughout the notation of Lévy-Leblond's paper (1971): the element of $\mathcal{G}_0$ denoted by $(b,\mathbf{a},\mathbf{v},R)$ will correspond to the transformation of $\mathbb{R}^4$, $\mathbf{x}'=R\mathbf{x}+\mathbf{v}t+\mathbf{a}$, $t'=t+b$. The factor group $\mathcal{G}/\mathcal{G}_0$ is Klein's group $V=\pi_0(\mathcal{G})$, generated by space inversion I_S and time inversion I_T. Moreover, $\mathcal{G}$ is a semidirect product group $\mathcal{G}=\mathcal{G}_0 \odot V$, the action of V on $\mathcal{G}_0$ being

$$I_S\colon (b,\mathbf{a},\mathbf{v},R)\to(b,-\mathbf{a},-\mathbf{v},R), \qquad I_T\colon (b,\mathbf{a},\mathbf{v},R)\to(-b,\mathbf{a},-\mathbf{v},R)$$

What can we tell about the candidates for the unitary subgroup? It must be a closed subgroup of index 1 or 2, including the connected component $\mathcal{G}_0$: there are four possible choices

(i) $\mathcal{G}_U=\mathcal{G}$
(ii) $\mathcal{G}_U=\mathcal{G}_0\cup I_S\mathcal{G}_0$
(iii) $\mathcal{G}_U=\mathcal{G}_0\cup I_T\mathcal{G}_0$
(iv) $\mathcal{G}_U=\mathcal{G}_0\cup I_{ST}\mathcal{G}_0$

Before making up our mind, we begin by studying the second cohomology group $H^2(SO_3(\mathbb{R}),\mathbf{T})$ of the unimodular orthogonal group in three dimensions. It is a compact and connected Lie group; in this case (Moore, 1964, Proposition 2.1), the second cohomology group is the group of two elements, C_2. Its elements will be denoted by $[l]$ with $l=\pm 1$. An arbitrary but fixed lifting of the class $[-1]$ taking the values ± 1 will be denoted ζ_{-1}.

The second cohomology group of $\mathcal{G}_0$ is $H^2(\mathcal{G}_0,\mathbf{T})=\mathbb{R}\otimes C_2$ (Bargmann, 1954; Brennich, 1970). The element $[M,l]$ will be the class of the factor system

$$\xi_{M,l}((b',\mathbf{a}',\mathbf{v}',R'),(b,\mathbf{a},\mathbf{v},R))=\zeta_l(R',R)\exp\left[iM\left(\tfrac{1}{2}b\mathbf{v}'^2+\mathbf{v}'R'\mathbf{a}\right)\right]$$

The group $\mathcal{G}$ is a semidirect product group and then the method of finding $H^2{}_*(\mathcal{G},\mathbf{T})$ indicated in the preceding section may be used. In the case we are dealing with, once a factor system $\xi\in Z^2(\mathcal{G}_0,\mathbf{T})$ has been chosen, is there any Borel function Λ: $\mathcal{G}\to\mathbf{T}$ satisfying the relation (2.4a)?

The answer is as follows: If the test of Theorem 3 is used, the factor systems $\tau_{I_S}\xi$, $\tau_{I_T}\xi$, and $\tau_{I_{ST}}\xi$ are given by

$$(\tau_{I_S}\xi)(g',g)=\xi(g',g)$$

$$(\tau_{I_T}\xi)(g',g)=[\xi(g',g)]^*$$

$$(\tau_{I_{ST}}\xi)(g',g)=[\xi(g',g)]^*$$

that is to say, $\bar{\tau}_{I_S}[M,l]=[M,l]$, $\bar{\tau}_{I_T}[M,l]=[-M,l]$, and $\bar{\tau}_{I_{ST}}[M,l]=[-M,l]$.

A direct use of Theorem 3 shows that the choices $\mathcal{G}_U=\mathcal{G}$, $\mathcal{G}_U=\mathcal{G}_0\cup I_T\mathcal{G}_0$, and $\mathcal{G}_U=\mathcal{G}_0\cup I_{ST}\mathcal{G}_0$ must be put away from our mind because they would only permit factor systems where the $\mathcal{G}_0$ part have $M=0$; if any of such choices is done, "massive systems" will not arise.

4. SPACE AND TIME INVERSION IN KINEMATIC GROUPS

In the case of Poincaré group the former test gives us no information. In fact, $H^2(\mathcal{P}_0,\mathrm{T})=C_2$; the factor systems are given by

$$\xi_l((b',\mathbf{a}',\mathbf{v}',R'),(b,\mathbf{a},\mathbf{v},R))=\zeta_l(R',R)$$

and the action of $V=\pi_0(\mathcal{P})$ on $\mathcal{P}_0$, α: $(b,\mathbf{a},\mathbf{v},R)\to(b^\alpha,\mathbf{a}^\alpha,\mathbf{v}^\alpha,R^\alpha)$, is such that $R^\alpha=R$, and therefore ξ_l is real and $\tau_\alpha\xi_l=\xi_l$ $\forall\alpha\in V$.

What can we tell about space reflection and time reversal in other kinematic groups (Bacry and Lévy-Leblond, 1968)? There are two kinds of kinematic groups: those of "relative time" such as de Sitter ($\mathbb{S}_\pm$), Poincaré ($\mathcal{P}$), para-Poincaré ($\mathcal{P}'$), inhomogeneous orthogonal four-dimensional [$IO(\mathbb{R},4)$], and Carrol ($\mathcal{C}$) groups, and those of absolute time such as Newton–Hooke (N), Galilean ($\mathcal{G}$), para-Galilean ($\mathcal{G}'$), and static groups ($\mathfrak{S}$). The second cohomology group $H^2(G_0,\mathrm{T})$ depends on this classification: if G_0 is a connected "relative time" kinematic group, $H^2(G_0,\mathrm{T})=C_2$, while if it is a connected "absolute time" kinematic group, $H^2(G_0,\mathrm{T})=\mathbb{R}\otimes C_2$.

If the elements of the kinematic groups are denoted (b,a,v,R), a lifting of the factor system in each case is given by

$$\xi_l((b',\mathbf{a}',\mathbf{v}',R'),(b,\mathbf{a},\mathbf{v},R))=\zeta_l(R',R) \qquad \text{if } G \text{ is a relative time group}$$

$$\xi_{[M,l]}((b',\mathbf{a}',\mathbf{v}',R'),(b,\mathbf{a},\mathbf{v},R))=\exp\left[iMf_{G_0}(g',g)\right]\cdot\zeta_l(R',R)$$

if G is an absolute time group

where f_G is the corresponding factor displayed in the following table:

	$f_{G_0}((b',\mathbf{a}',\mathbf{v}',R'),(b,\mathbf{a},\mathbf{v},R))$
$N_\pm$	$\pm\frac{1}{2}\left[\frac{\mathbf{a}'^2}{\tau}+\tau\mathbf{v}'^2\right]\sin_\pm\frac{b}{\tau}\cos_\pm\frac{b}{\tau}\pm\mathbf{a}'\mathbf{v}'\sin^2_\pm\frac{b}{\tau}+\mathbf{v}'R'\mathbf{a}\cos_\pm\frac{b}{\tau}\pm\mathbf{a}'R'\mathbf{a}\frac{1}{\tau}\sin_\pm\frac{b}{\tau}$
$\mathcal{G}'$	$\frac{1}{2}b\mathbf{a}'^2+\mathbf{a}'R'\mathbf{v}$
$\mathcal{G}$	$\frac{1}{2}b\mathbf{v}'^2+\mathbf{v}'R'\mathbf{a}$
$\mathcal{S}$	$\mathbf{v}'R'\mathbf{a}$

These explicit forms of the factor systems show that in an "absolute time" kinematic group the only possible choice for the unitary subgroup is $G_U=G_0\cup I_SG_0$, such as in the Galilean case. On the other hand, in the case of a "relative time" kinematic group, the theorems we have considered will give no information about the unitary subgroup. If the choice $G_U=G_0\cup I_SG_0$ is made, the second cohomology groups which are necessary in the search for a representation group are, respectively,

$$H^2{}_*(G,\mathbf{T})=C_2\otimes(C_2\otimes C_2)\qquad \text{if } G \text{ is a relative time group}$$

$$H^2{}_*(G,\mathbf{T})=(\mathbb{R}\otimes C_2)\otimes(C_2\otimes C_2)\qquad \text{if } G \text{ is an absolute time group}$$

as a direct application of Theorem 1 and a further study of the equivalence of factor systems fairly quickly show.

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